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MAP TRACKING INCLUDING THE EFFECT OF STOCHASTIC RADIATION

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Abstract

Using transfer maps to simulate charged particle motion in accelerators is advantageous since it is much faster than tracking step-by-step. One challenge to using transfer maps is to properly include radiation effects. The effect of radiation can be divided into deterministic and stochastic parts. While computation of the deterministic effect has been previously reported, handling of the stochastic part has not.

In this paper, an algorithm for including the stochastic effect is presented including taking into account the finite opening angle of the emitted photons. A comparison demonstrates the utility of this approach. Generating maps which include radiation has been implemented in the PTC software library which is interfaced to the Bmad toolkit.

INTRODUCTION

Particle in accelerators tracking is an important and widely used simulation tool since it is the only reliable technique that can accurately and reliably probe the nonlinear effects, such as particle loss, that can develop in particle beams over many turns [1]. Routinely, particle tracking is done either by tracking step-by-step which is slow, or by using one or more transfer maps which is fast but potentially inaccurate.

Radiation effects, when relevant, complicate tracking. While one can include the radiation effects into step-by-step tracking, this is generally not done when using transfer maps or is done using a simple energy kick at the end of a map which in many cases is not accurate enough. In this paper we show how to incorporate the radiation effect with map tracking.

The reaction of a particle due to the emission of a photon can be modeled as the sum of two kicks: There is the "deterministic" kick which is the average radiation emitted at the emission point plus a "stochastic" kick which represents the fluctuations around the average. If the effect on a particle via emission of a photon is small, which is generally the case in any practical machine, the stochastic kick can be modeled as having Gaussian probability profile [2].

Inclusion of the deterministic part has been previously reported [3, 4], handling of the stochastic part has not and this is the subject of this paper. Comparisons of map tracking with element-by-element tracking demonstrate the utility of this approach.

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RADIATION EMISSION

The phase space coordinate system used for the analysis is (x, p_x, y, p_y, z, p_z) where

$$p_{x,y} = \frac{P_{x,y}}{P_0}, \quad z = -\beta c(t - t_0), \quad p_z = \frac{P - P_0}{P_0} \tag{1}$$

with $P_{x,y}$ being the transverse momentum, P is the momentum, P_0 is the reference momentum, βc the particle velocity, t the time, and t_0 the reference time. It will be assumed that the particle energy is large enough so that p_z can be approximated by $(E - E_0)/E_0$ where E is the particle total energy and E_0 is the reference energy. If this approximation is not valid, it is fairly straightforward to extend the results here to lower energies.

The energy loss ΔE over some small path length L_p is modeled via [2]

$$\frac{\Delta E}{E_0} = -k_E \equiv -\left[k_d \left\langle g^2 \right\rangle L_p + \sqrt{k_f \left\langle g^3 \right\rangle L_p} \; \xi_E \right] p_r^2 \tag{2}$$

where $p_r \equiv 1 + p_z$, g is the bending strength ($1/g$ is the orbital bending radius), ξ_E is a Gaussian distributed random number with unit sigma and zero mean, and $\langle \ldots \rangle$ is an average over the path length. In the above equation, the deterministic k_d and stochastic k_f coefficients are given by

$$k_d = \frac{2\, r_c}{3}\, \gamma_0^3, \qquad k_f = \frac{55\, r_c\, \hbar}{24\sqrt{3}\, m\, c}\, \gamma_0^5 \tag{3}$$

where γ_0 is the energy factor of an on-energy particle and r_c is the particles "classical radius" given by $r_c = q^2/4\,\pi\,\epsilon_0\, m\, c^2$ where q is the particle's charge and m is the particle's mass.

Ignoring the finite opening angle for now, radiation emitted in the forward direction preserves the angular orientation of the particle's motion which leads to the following equations for the changes in the momentum phase space coordinates

$$(\Delta p_x, \Delta p_y) = -\frac{k_E}{p_r}\,(p_x, p_y), \quad \Delta p_z \approx \frac{\Delta E}{E_0} = -k_E\,. \tag{4}$$

The fact that an emitted photon is not exactly collinear with the particle direction (often called the "opening angle") can be modeled as a separate process from the energy loss. The change $\Delta p_\perp$ in the momentum transverse to the bending plane is given by

$$\Delta p_\perp = \sqrt{k_v \left\langle g^3 \right\rangle L_p}\; \xi_v \tag{5}$$

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where the ξ_v is a Gaussian distributed random number with unit sigma and zero mean and is independent of the ξ_E in Eq. (2). The opening angle coefficient k_v is given by

$$k_v = \frac{13\, r_c\, \hbar}{24\sqrt{3}\, m\, c}\, \gamma_0^3 \, . \tag{6}$$

TRANSPORT MAP WITH RADIATION INCLUDED

The transport maps considered here are with respect to a reference orbit which is the closed orbit for lattices with a closed geometry and for lattices with an open geometry the reference orbit is the beam orbit which has some given initial position. In both cases, the reference orbit must be calculated including radiation damping but ignoring the stochastic effects. Since the stochastic kick is a random walk in six dimensions, the transfer map from position s_1 to position s_2 will be of the form

$$\delta\mathbf{r}_2 = \mathcal{M}_{21}(\delta\mathbf{r}_1) + \mathcal{S}_{21}\,\mathbf{\Xi} \tag{7}$$

where $\delta\mathbf{r}_1$ and $\delta\mathbf{r}_2$ are the particle positions with respect to the reference orbit at s_1 and s_2 respectively, and $\mathcal{M}_{21}$ is the transfer map with damping. Since $\mathcal{M}_{21}$ is computed with respect to the beam centroid orbit, there is no constant part to the map. The stochastic radiation part in the above equation is represented by a 6×6 matrix $\mathcal{S}$ times a 6-vector

$$\mathbf{\Xi} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \tag{8}$$

with each ξ_i being an independent Gaussian distributed random number with unit sigma and zero mean. The stochastic transport is treated here only in lowest order. This is a good approximation as long as the radiation emitted is small enough in the region between s_1 and s_2. This is true in nearly all practical cases. If this approximation is violated, the beam will have a non-Gaussian shape.

MAP WITH DETERMINISTIC DAMPING

The transfer map with damping $\mathcal{M}$ is calculated by adding in the effect of the damping when integrating the equations of motion to form the map. This has been discussed by Nishikawa [5], Ohmi [3] and Chao [4]. Through a given lattice element, it is generally very safe to assume that the change in energy is small compared to the energy of a particle. Thus the matrix $\mathbf{M}$, which is the first order part of $\mathcal{M}$, through an element can is computed via first order perturbation theory to be

$$\mathbf{M} = \mathbf{T} + \mathbf{Z} \tag{9}$$

where $\mathbf{T}$ is the transfer matrix without damping calculated around the reference orbit and $\mathbf{Z}$ is the change in $\mathbf{T}$ due to damping computed via

$$\mathbf{Z} = \int_{s_1}^{s_2} ds\, \mathbf{T}_{2,s}\, \mathbf{d}(s)\, \mathbf{T}_{s,1} \tag{10}$$

where the local damping matrix $\mathbf{d}$ is computed from Eqs. (2) and (4)

$$\mathbf{d} = -\,k_d \times \tag{11}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{dg^2}{dx} p_x p_r & g^2 p_r & \frac{dg^2}{dy} p_x p_r & 0 & 0 & g^2 p_x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{dg^2}{dx} p_y p_r & 0 & \frac{dg^2}{dy} p_y p_r & g^2 p_r & 0 & g^2 p_y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{dg^2}{dx} p_r^2 & 0 & \frac{dg^2}{dy} p_r^2 & 0 & 0 & 2g^2 p_r \end{pmatrix} .$$

All quantities are evaluated on the closed orbit.

STOCHASTIC TRANSPORT

The $\mathcal{S}$ matrix in Eq. (7) is calculated by first noting that, to linear order, the distribution of $\delta\mathbf{r}_2$ due to stochastic radiation over some length ds as some point s is

$$\delta\mathbf{r}_2 = \sqrt{ds}\, \mathbf{M}_{2,s}\, \left(\mathbf{F}_f(s)\, \xi_E + \mathbf{F}_v\, \xi_v\right) \tag{12}$$

where $\mathbf{M}_{2,s}$ is the first order part (matrix) of the map $\mathcal{M}_{2,s}$ from s to s_2, and $\mathbf{F}_f$ and $\mathbf{F}_v$ are derived from Eqs. (4) and (5)

$$\mathbf{F}_f = \sqrt{k_f\, g_0^3}\, (0, p_x\, p_r, 0, p_y\, p_r, 0, p_r^2) \tag{13}$$
$$\mathbf{F}_v = \sqrt{k_v\, g_0}\, (0, -g_y, 0, g_x, 0, 0) \tag{14}$$

where k_f, p_x, p_y and p_z are to be evaluated on the reference orbit and $\mathbf{g} = (g_x, g_y)$ is the curvature vector with $|\mathbf{g}| = g$ and which points away from the center of curvature of the particle's orbit. Since $\delta\mathbf{r}$ is, by definition, the deviation from the reference orbit, $p_x = r_2$ and $p_y = r_4$ will be zero on the reference orbit. The covariance matrix $\boldsymbol{\sigma}_\gamma$ is defined by $\sigma_{\gamma ij} \equiv \langle r_i\, r_j \rangle_\gamma$ where $\langle \ldots \rangle_\gamma$ is an average over the photon emission spectrum. The contribution, $\boldsymbol{\sigma}_{\gamma 21}$, to the covariance matrix at s_2 due to the stochastic emission over the region between s_1 and s_2, is

$$\boldsymbol{\sigma}_{\gamma 21} = \int_{s_1}^{s_2} ds\, \mathbf{M}_{2,s} \left[\mathbf{F}_f(s)\, \mathbf{F}_f^t(s) + \mathbf{F}_v(s)\, \mathbf{F}_v^t(s)\right] \mathbf{M}_{2,s}^t \tag{15}$$

where the t superscript indicates transpose. From Eq. (12) it is seen that $\boldsymbol{\sigma}_{\gamma 21}$ is related to $\mathcal{S}$ via

$$\boldsymbol{\sigma}_{\gamma 21} = \mathcal{S}_{21}\, \mathcal{S}_{21}^t \, . \tag{16}$$

The calculation of $\mathcal{S}_{21}$ involves calculating $\boldsymbol{\sigma}_{\gamma 21}$ via Eq. (15) and then using Eq. (16) to solve for $\mathcal{S}_{21}$. While Eq. (16) does not have a unique solution, any matrix $\mathcal{S}$ that satisfies Eq. (16) will give the correct distribution $\boldsymbol{\sigma}_{\gamma 21}$. A good choice for constructing $\mathcal{S}$ is a Cholesky decomposition with can be done efficiently and is robust.

EMITTANCE CALCULATION

As a side note, the beam emittance can be calculated [3] by expressing the covariance matrix $\boldsymbol{\sigma}_\gamma$ at s_2 relative to the covariance matrix at s_1

$$\boldsymbol{\sigma}_\gamma(s_2) = \boldsymbol{\sigma}_{\gamma 21} + \mathbf{M}_{21}\, \boldsymbol{\sigma}_\gamma(s_1)\, \mathbf{M}_{21}^t \, . \tag{17}$$

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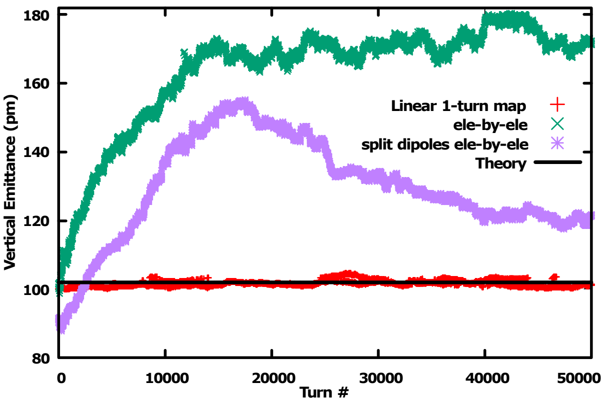

Figure 1: Tracking results in a low emittance lattice. The black line is theory, the red points are from tracking with a linear 1-turn map (Eq. (7)). The green points are from tracking element-by-element with a radiation kick put in at element ends. The purple points are from element-by-element tracking like the green tracking except here all dipoles have been split in half.

The beam size matrix σ is not the same as the covariance matrix since the beam size matrix is an average over the particles of a beam and not an average over the photon emission spectrum. However, in equilibrium, the two are the same. Setting $s_1 = s_2 = s$, Eq. (17) becomes a linear equation in the unknown elements of $\boldsymbol{\sigma}_\gamma$ and is easily solved. Once the beam size matrix is known, the emittances can be extracted using an algorithm given by Wolski [6].

EXAMPLE TRACKING SIMULATION

Figure 1 shows an example of how modeling the stochastic radiation kick can greatly improve tracking accuracy. A low emittance electron lattice for the ALS-U ring at LBL was used to track a bunch of particles for 50,000 turns. The horizontal black line in the figure marks the theoretical emittance for the vertical mode which was 102 pm. The theoretical emittance was calculated using the algorithm as discussed above. This calculation was cross-checked with a calculation using radiation integrals [2] and good agreement was found. In this case, the emittance increase from the vertical opening angle was small and had a negligible effect on the simulations

The red, green and purple points in Fig. 1 show the emittance as calculated from particle tracking. On any given turn, the emittance of the beam is computed by first computing the beam sigma matrix and then using Wolski's [6] algorithm to extract the emittances. The transverse damping time is about 10,000 turns so the beam was tracked for about five damping times.

With the red points, the tracking used a linear one-turn map of the form Eq. (7). As can be seen, the emittance of the tracked beam matches well the theoretical emittance. The speedup of map tracking over element-by-element tracking with this lattice of some 1100 elements was approximately a factor of 500. Even using a fifth order map the speed increase is about a factor of 20. This shows the power of tracking with a map. One downside of map tracking is the time it takes to construct the map in the first place but this is more than balanced by the speed of tracking particles.

The green points are from tracking element-by-element with a radiation kick (Eqs. (4) and (5)) applied at the ends of all elements. With many lattices, lumping the radiation kick to be at the element ends is a good approximation. However, with this low emittance lattice, the result is that the emittance equilibrates at a value that is about 70% higher than the theoretical value. This is due to the design of the lattice where the dispersion has been constructed to be significantly smaller in the middle of bending magnets as opposed to the ends. Since the effect of the stochastic emission on the emittance scales as the square of the dispersion, only radiating at the dipole ends leads to an increased simulated beam size. For comparison, all the dipoles were split in the middle and the tracking redone as shown by the purple points. With the increased number of emission points, the beam equilibrium emittance is closer to the theoretical being only about 20% higher. The lattice used here had some 228 dipoles which means that with the split dipoles there were 684 bend radiation points. This is to be compared to the map tracking which only had a single six-dimensional stochastic kick per turn but was more accurate.

CONCLUSION

How to incorporate the stochastic radiation kick with map tracking has been presented. This includes taking into account the finite opening angle of the photons. Tracking simulations with a low emittance electron lattice shows that a one-turn map which has a single six-dimensional stochastic kick can outperform tracking where there are hundreds of radiation kicks per turn.

The stochastic kick in the map with radiation (Eq. (7)) represents the lowest order contribution. This is the same approximation used by the radiation integrals analysis and the emittance calculation as outlined above. Extending the analysis to higher order is conceptually straightforward. Essentially, the **M** matrices in Eq. (15) become maps and the term in square brackets acquires a dependence upon the transverse coordinates. In this case, the elements of the $\boldsymbol{\sigma}_\gamma$ matrix are functions of the particle coordinates $\delta\mathbf{r}_1$ and thus $\boldsymbol{\sigma}_\gamma$ and the Cholesky decomposition must be evaluated particle-by-particle when applying the map.

Creation of Taylor maps of arbitrary order that include radiation effects including the finite photon opening angle have been incorporated [5] into the PTC library [7] and this has been interfaced to the Bmad [8] toolkit. The maps are partially inverted (implicit) to achieve symplecticity when there is no radiation damping. The maps also include spin transport.

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