Study of Nonlinear Dynamics in the 4-D Hénon Map Using the Square Matrix Method and Iterative Methods

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The iterative method for finding particle frequencies and orbits has been expanded to 4-D systems. Combining it with the Square Matrix method shows promising results in its ability to be used to study the resonance structure of a phase space and to find the dynamic aperture.

Iteration Method

Representing $z_{x,y}=\exp(i\theta_{x,y})$ this process finds a diffeomorphism to a rigid rotation (h and g).

Want to find

$$\theta_x = \alpha + h(\alpha,\beta)$$
$$\theta_y = \alpha + g(\alpha,\beta)$$

Form of the Solution

$$\hat{d}^{(n+1)}(\alpha,\beta) = \frac{\hat{\eta}^{(n)}_{j,nm}}{e^{im\rho_x^{(n+1)}+in\rho_y^{(n+1)}} - 1}$$

Iteration to Solve

$$d^{(n+1)}(\alpha+\rho_x^{(n+1)},\beta+\rho_y^{(n+1)}) - d^{(n+1)}(\alpha,\beta) = f_j(\alpha,\beta,h^{(n)}(\alpha,\beta),g^{(n)}(\alpha,\beta)) - \rho_j^{(n+1)}$$

Note: f is the change in θ for one turn, d is either h or g and j is x or y accordingly, η is the RHS of the iteration, and a hat denotes it's a Fourier coefficient of that function

4-D Hénon Map

Represents a linear lattice with a single sextupole kick

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{n+1} = \begin{pmatrix} \mathbf{R}(\mu_x) & 0 \\ 0 & \mathbf{R}(\mu_y) \end{pmatrix} \begin{pmatrix} x \\ p_x - x^2 + y^2 \\ y \\ p_y + 2xy \end{pmatrix}_n$$

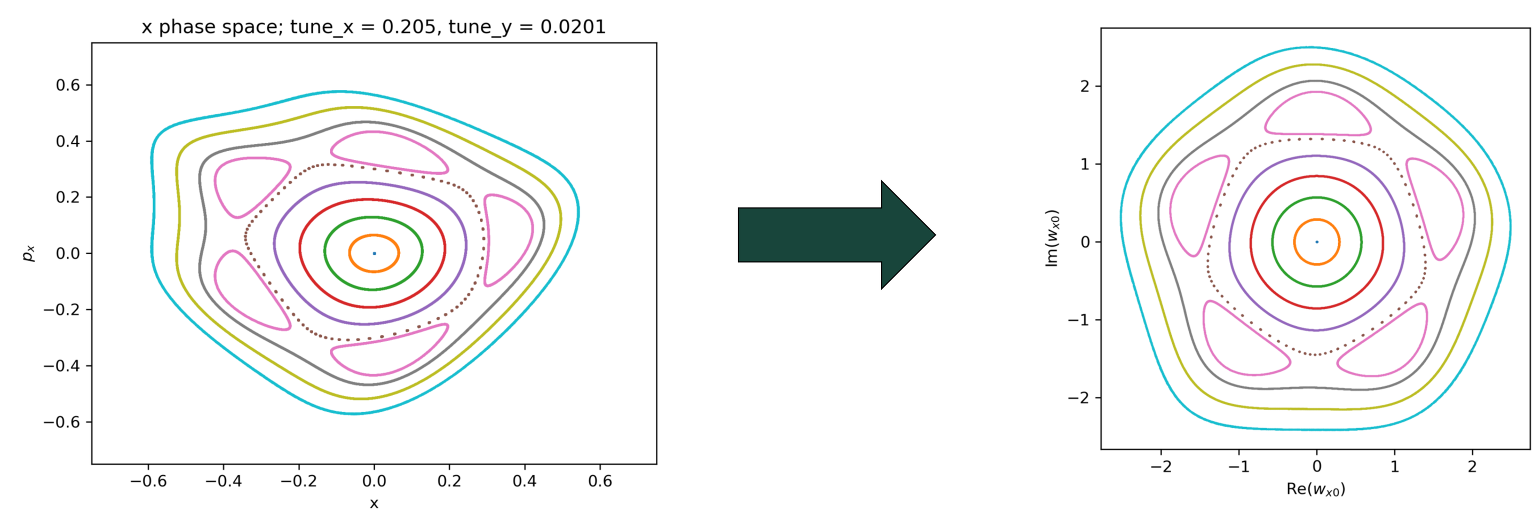
Representation in z form where $z_x=x-ip_x$ and $z_y=y-ip_y$ where z' is z after one turn

$$z_x' = \frac{e^{i\mu_x}}{4}\left(-i(z_x^*)^2 - 2iz_x^*z_y + i(z_y^*)^2 + 2iz_y^*z_y - iz_x^2 + 4z_x + iz_x^2\right)$$

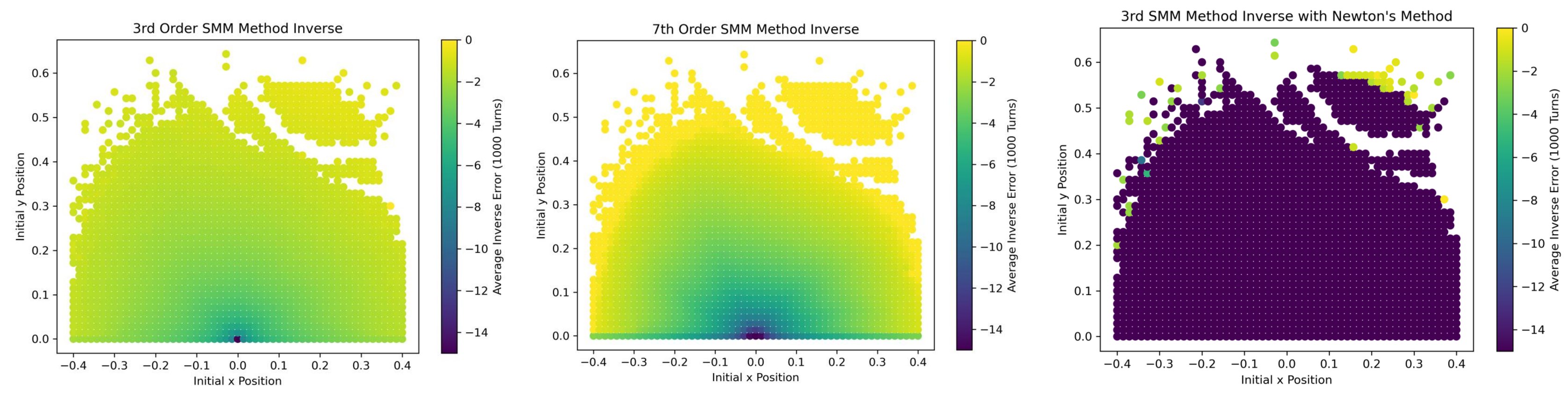
$$z_y' = \frac{e^{i\mu_y}}{2}\left(-iz_x^*z_y^* + iz_x^*z_y + iz_y^*z_x + iz_xz_y + 2z_y\right)$$

Square Matrix Method

By using a Jordan decomposition on a matrix of truncated higher order terms of the map we can get a transformed phase space that is closer to a pure rotation.

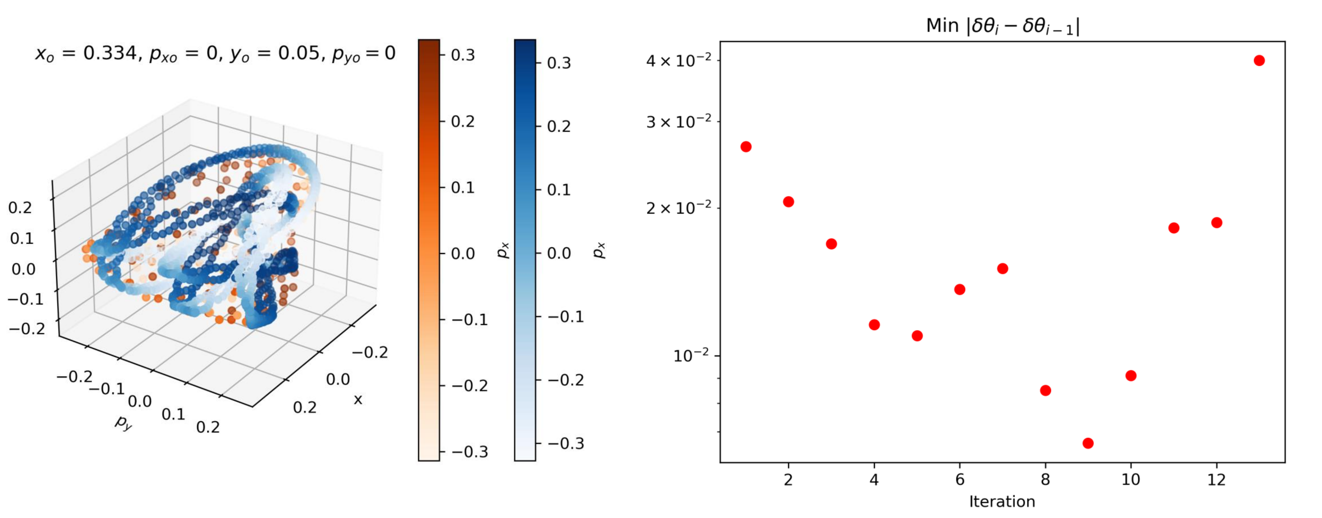


SMM Inverse

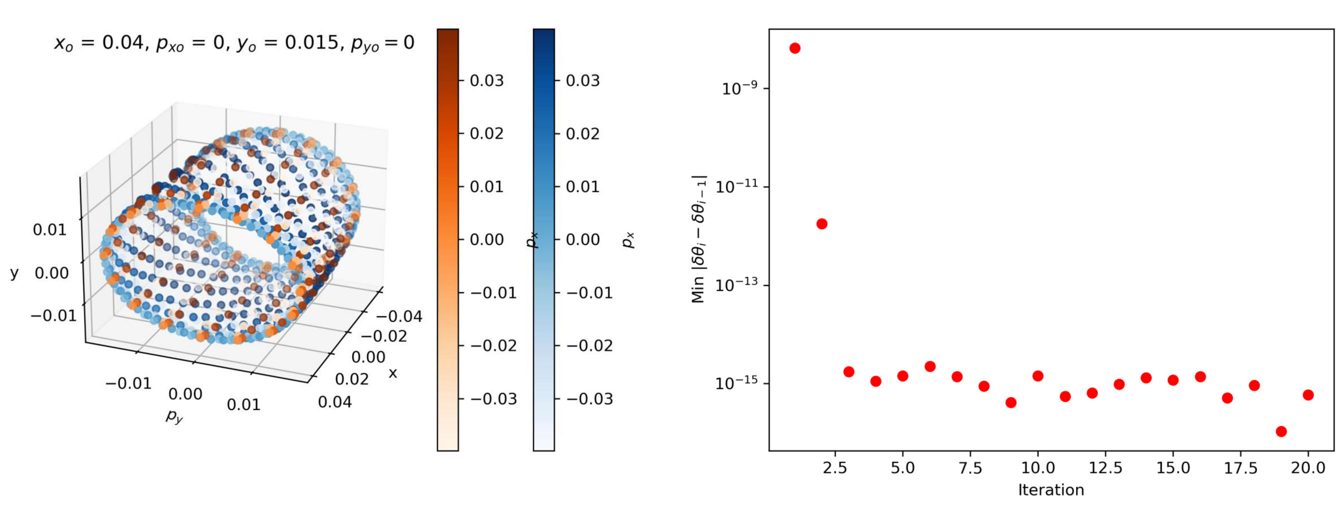


Averaged inverse error for the first 1000 turns using just the 3rd order and 7th order square matrix method and adding Newton's method.

Single Particle Example using Iterative Method: ν_x=0.282, ν_y=0.6135



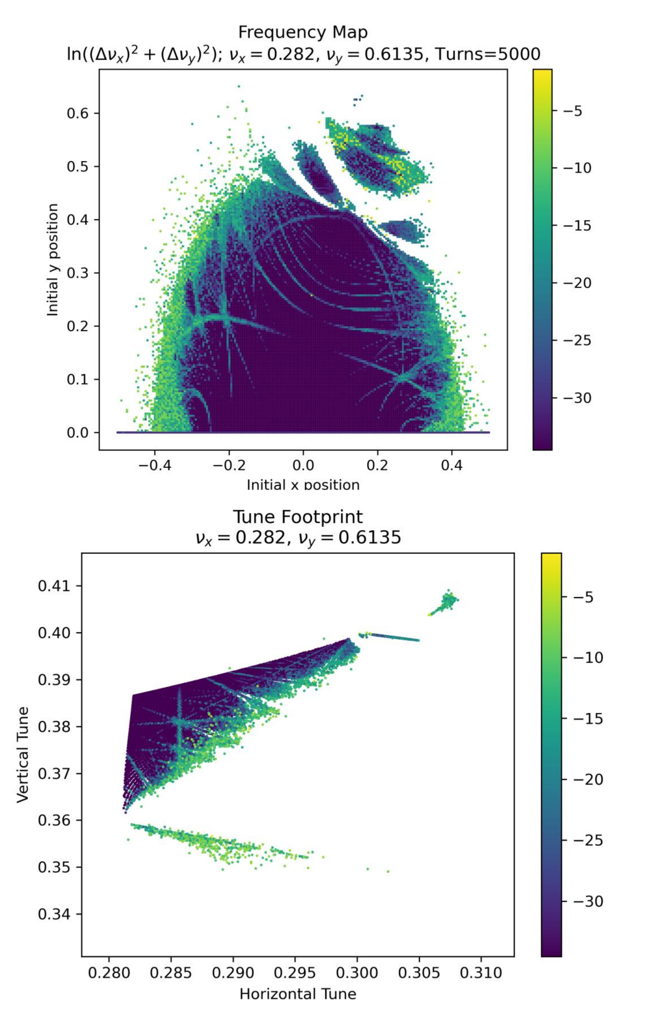
The iterative method fails to converge to a value.



The iterative method converges to ν_x = 0.28202878 and ν_y = 0.38629198 which differ from NAFF by 10^{-9} and 10^{-10} respectively

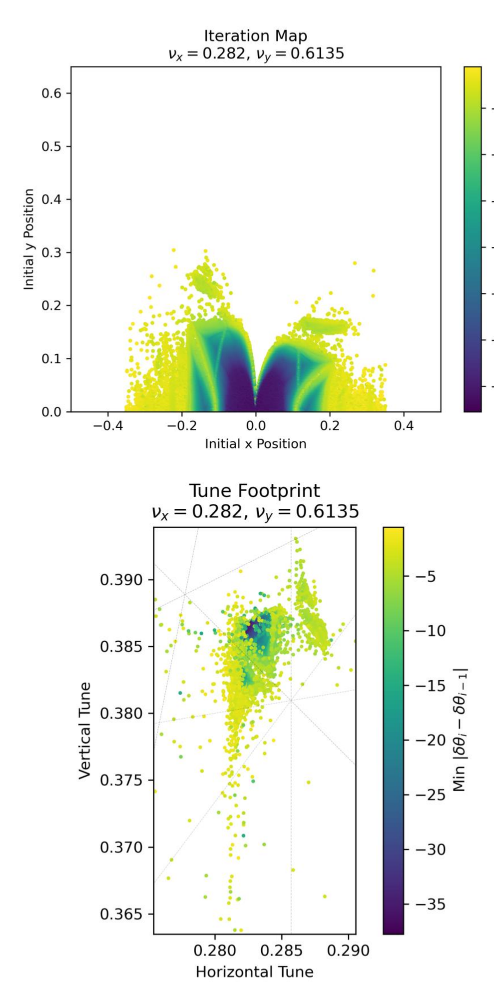
Iteration Results

Frequency Map



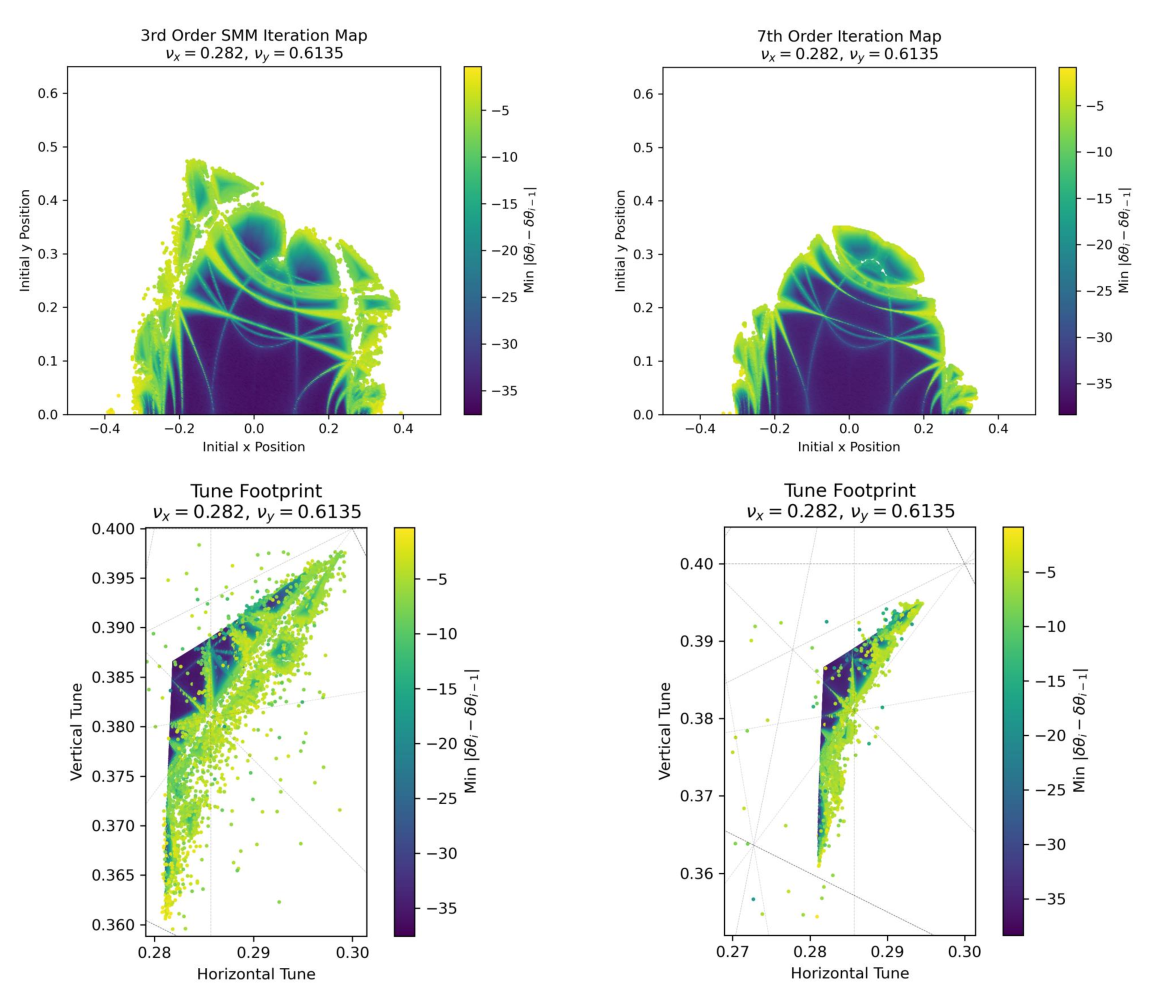
Frequency Map and Tune Footprint generated from the NAFF algorithm [3]

Iteration Map



Iteration Map and Tune Footprint generated from the iteration method

Iteration Map with the Square Matrix method



Iteration Maps and Tune Footprints generated from the Iteration method using the Square Matrix method

Conclusion

- Using the Square Matrix method increases the area of convergence for the iteration method.
- The iteration map shows similar features compared to the frequency map such as the resonance structures
- The iterative method proves to be a promising avenue for studying the resonance structure of a system

Future Work

- Further research is needed to extend this area and extended the method to a 6-D phase space and so it can be used for dynamic aperture measurement

References and Poster

Acknowledgements

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